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NOTES
ON
DYNAMICS OF ROTATION

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1. *Linear velocity.*

Uniform when no force acts ; variable, when force acts.

2. Statement of the parallelogram of velocities. Illustrations of this principle :

(*a*) Man rowing uniformly across a stream, which flows between parallel banks with uniform or variable velocity.

(*b*) Motion of a projectile, whose initial velocity is 200 feet a second, inclined at an angle of 45° to the horizon.

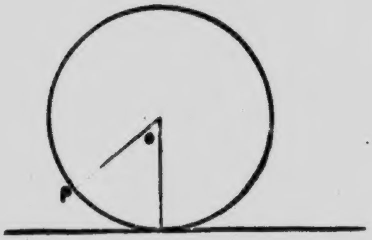
3. *Angular velocity.*

When a point moves uniformly in a circle, $v = a\omega$.

4. A wheel rolls uniformly along a horizontal plane : shew that, at any instant,

(*a*) the velocity of the centre of the wheel is $a\omega$,
(*b*) the velocity of the highest point is $2a\omega$,
(*c*) the velocity of the lowest point is zero,
(*d*) the velocity of the point *P* in the figure is

$$2a\omega \sin\frac{\theta}{2}.$$



N.B.—Use the principle of the parallelogram of velocities.

5. *Centre of gravity.*

The point in a body at which the weight may be supposed to act.

6. Assuming the centre of gravity of a uniform rod to be at the centre, it follows that the centre of gravity of all symmetrical bodies is at the centre of symmetry ; where bodies are symmetrical in respect of a line or plane, the centre of gravity must lie in that line or plane.

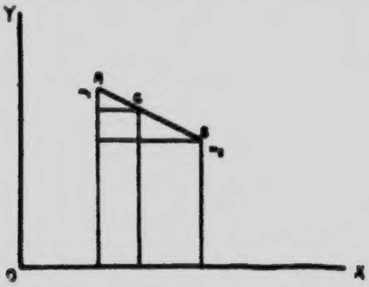
7. *C.G.* of a triangular area : deduce from rod. Found also by experiment.

8. *C.G.* of any body, more particularly if in the form of a flat plate, found by experiment.

9. New conception of *centre of gravity*: the point in a body about which rotation takes place most easily.

Any axis passing through the centre of gravity of a body may therefore be viewed as an *axis of least effort*.

10. General method of finding the centre of gravity of a body or system of bodies.



Let two small masses, m_1, m_2, be at the points A, B: G, their centre of gravity is given by the relation

$$m_1 \times AG = m_2 \times BG.$$

Hence

$$m_1 (\bar{x} - x_1) = m_2 (x_2 - \bar{x})$$

the coördinates of A, B, G being, respectively, x_1, y_1; x_2, y_2; $\bar{x}, \bar{y}$. Therefore

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

Similarly, it may be shewn that

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}.$$

If there are three small masses m_1, m_2, m_3 at the points x_1, y_1; x_2, y_2; x_3, y_3; then, replacing m_1 and m_2 by a mass $m_1 + m_2$ at G, G', the centre of gravity of the three masses m_1, m_2, m_3 will be defined by the relation

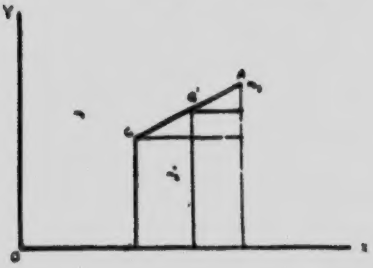
$$(m_1 + m_2)\, GG' = m_3 \times AG'$$

$$\therefore (m_1 + m_2)\left\{\bar{x} - \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}\right\} = m_3 (x_3 - \bar{x})$$

$$\therefore \bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}.$$

Similarly,

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} .$$



Therefore, generally, it may be shewn that for any number of masses $m_1, m_2, \ldots m_n$, the centre of gravity is defined by the relations

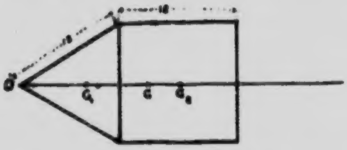
$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \ldots \text{etc.}}{m_1 + m_2 + \ldots \text{etc.}} = \frac{\Sigma(m\,x)}{\Sigma(m)} ,$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + \ldots + \text{etc.}}{m_1 + m_2 + \ldots + \text{etc.}} = \frac{\Sigma(m\,y)}{\Sigma(m)} .$$

These expressions are true in the case where $m_1, m_2, \ldots$ are infinitely small, and also in the case where they represent finite masses collected at individual centres of gravity.

11. *Examples.*

(*a*) To find the centre of gravity of a plate in the form of the figure shewn : a square joined to an equilateral triangle.

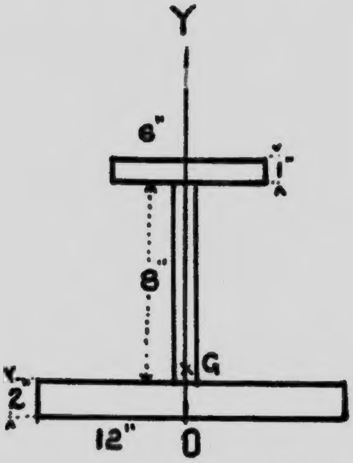


The centre of gravity must lie on *O X*, the line of symmetry. Collect the mass of the triangle at G_1 and the mass of the square at G_2. Then

$$\bar{x} = OG = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} =$$

$$\frac{36\sqrt{3} \times 4\sqrt{3} + 144 \times 6(1 + \sqrt{3})}{36\sqrt{3} + 144} = 13.4.$$

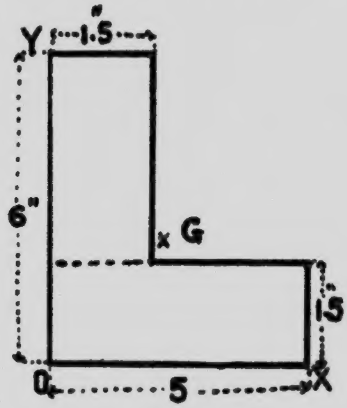
(*b*)



The centre of gravity of the section shewn here is on the line of symmetry *O Y*.

$$\overline{y} = OG = \frac{12 \times 2 \times 1 + 8 \times 1 \times 6 + 6 \times 1 \times 10.5}{12 \times 2 + 8 \times 1 + 6 \times 1} = 3''.55.$$

(*c*)



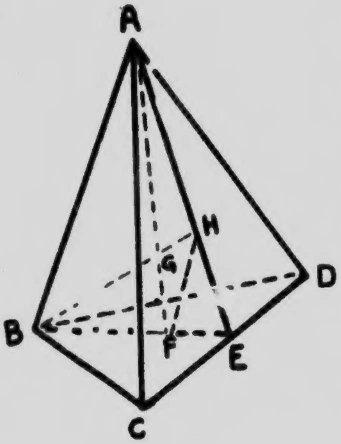
The centre of gravity of this section will be given by

$$\overline{x} = \frac{5 \times 1.5 \times 2.5 + 4.5 \times 1.5 \times .75}{5 \times 1.5 + 4.5 \times 1.5} = 1.67.$$

$$\overline{y} = \frac{5 \times 1.5 \times .75 + 4.5 \times 1.5 \times 3}{5 \times 1.5 + 4.5 \times 1.5} = 1.8.$$

The position of the point $\overline{x}$, $\overline{y}$ is shewn at *G*.

(*d*)



The centre of gravity of a tetrahedron $ABCD$ is determined by dividing it up into triangular plates parallel to BCD: the centre of gravity thus lying on AF, where F is the centre of gravity of BCD. Similarly the centre of gravity of the tetrahedron lies on BH where H is the centre of gravity of ACD: G is the required $C.G.$

$$\text{and } \frac{FH}{AB} = \frac{FG}{GA} = \frac{EF}{EB} = \tfrac{1}{3}$$

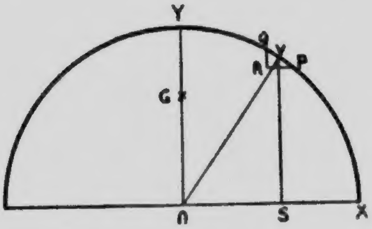
$$\therefore FG = \tfrac{1}{3}\, GA$$

$$\therefore FG = \tfrac{1}{4}\, FG$$

N.B.—Since any pyramid, or, in the limiting case, a cone, may be divided up into a large number of tetrahedra, it follows that the centre of gravity of any such pyramid or cone lies at a distance from the base one fourth of the whole distance from the apex to the base.

12. *Geometrical methods* of finding the centres of gravity of some figures bounded by curves.

(*a*) *A semicircular arc* (radius *a*).



G will evidently lie on the line of symmetry, OY. If PQ be a small element of the arc

$$\bar{y} = OG = \frac{\Sigma(my)}{\Sigma(m)} = \frac{\Sigma(PQ \times y)}{\pi a}$$

considering the density of the arc as unity.

$$\text{But } \cos RPQ = \frac{PR}{PQ}$$

$$\cos SVO = \frac{SV}{VO}$$

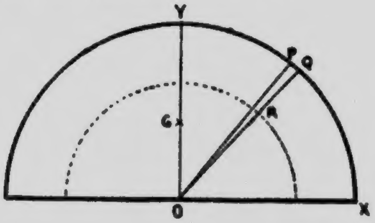
V being the centre of PQ.
And the angles RPQ, SVO are equal.

$$\therefore \frac{PR}{PQ} = \frac{SV}{VO} = \frac{y}{a}$$

$$\therefore PQ = \frac{a}{y} \times PR$$

$$\therefore \overline{y} = \frac{\Sigma\left(y \times \frac{a}{y} \times PR\right)}{\pi a} = \frac{a\,\Sigma(PR)}{\pi a} = \frac{2a^2}{\pi a} = \frac{2a}{\pi}$$

(*b*) Semi-circular plate (radius a).



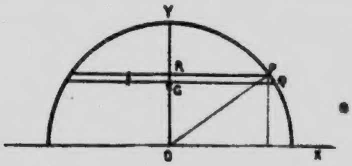
Dividing the circular plate up into an indefinite number of infinitely small triangular plates, such as OPQ, it is evident that these triangles may be replaced by their masses (all equal), collected at their centres of gravity.

The semi-circular plate may thus be replaced by a series of masses lying on a semi-circle of radius OR, equal to $\frac{2}{3}a$.

But by (*a*) the centre of gravity of this arrangement is given by

$$\overline{y} = OG = \frac{2(\frac{2}{3}a)}{\pi} = \frac{4a}{3\pi}$$

(*c*) A hollow hemispherical shell (radius a).



Divide the shell up into an infinite number of infinitely small annuli by planes parallel to the base, so that the distances cut off by OY are each equal to h.

Then the surface of each annulus shewn in the figure is equal to

$$2\pi \times PR \times PQ$$

$$\text{But } \frac{h}{PQ} = \frac{PR}{OP} \qquad \therefore\ PQ = h \times \frac{OP}{PR}$$

$$\therefore\ \text{surface} = 2\pi \times PR \times h \times \frac{OP}{PR} = 2\pi a h.$$

And this is constant.

Therefore the surface of the shell may be replaced by a number of equal small masses each collected at its centre of gravity, which lies on OY.

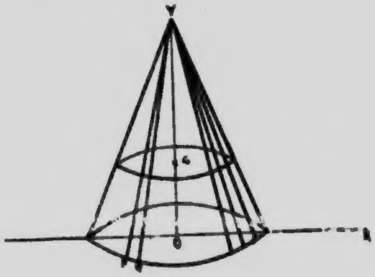
The centre of gravity of the shell is thus the same as that of a number of equal masses distributed uniformly along OY.

$$\therefore\ OG = \bar{y} = \frac{a}{2}$$

(*d*) Solid hemisphere (radius *a*).

The centre of gravity will lie on the axis of symmetry. Dividing the hemisphere up into an infinite number of infinitely small equal cones with a common apex at the centre of the hemisphere, it is evident that, since each cone has its centre of gravity at a distance $\frac{3}{4}a$ from the apex, the hemisphere may be replaced by a series of particles arranged on a hemispherical shell of radius $\frac{3}{4}a$. Therefore the centre of gravity of the hemisphere is $\frac{1}{2} \times \frac{3}{4}a$ or $\frac{3}{8}a$ from the centre.

(*e*) Hollow conical shell.

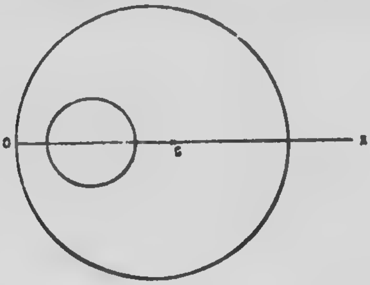


If the surface of the shell be divided into an indefinite number of infinitely small equal triangular areas, such as YPQ, the centre of gravity of each of which is $\frac{2}{3}$ the distance from Y to the rim of the cone, it will be seen that the centre of gravity of the whole surface is the same as that of a circle of equal masses, arranged symmetrically about OY. The centre of gravity of the shell is therefore at the centre of this circle, that is at *G*, such that

$$\overline{y} = OG = \tfrac{1}{3}h$$

if *h* be the height of the cone.

13. A circular area of radius *b* is cut from a circular area of radius *a*: to find the centre of gravity of the portion left.



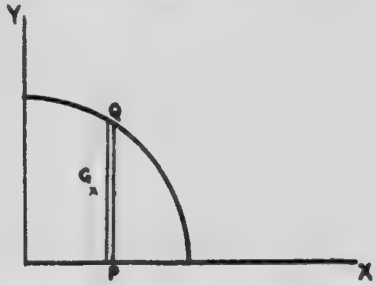
If the radius of the larger circle be *a*, of the smaller one *b*, and if the distance between their centres be *c*, then the centre of gravity of the portion required is at a point *G*, to the right of the centre of the larger circle, on the line of symmetry *OX*, such that

$$a = \frac{\pi b^2(a-c) + \pi(a^2-b^2)\cdot\overline{x}}{\pi a^2}$$

where $\overline{x} = OG$, and is therefore known.

14. Determination of the centre of gravity (or centroid) of a body by the Integral Calculus.

As an illustration of the method to be employed, take the quadrant of a circular area, radius *a*.



Divide the area up into infinitely small areas, such as *PQ*, each equal to $y\Delta x$: x, y, being the coordinates of any point *Q*.

$$\text{Then } y = \sqrt{a^2 - x^2}$$

$$\bar{x} = \frac{\Sigma(mx)}{\Sigma(m)} = \frac{\Sigma(y\Delta x \times x)}{\frac{1}{4}\pi a^2}$$

Taking the density as unity.

$$\therefore \bar{x} = \frac{\int_0^a xy\,dx}{\frac{1}{4}\pi a^2} = \frac{\int_0^a x\sqrt{a^2 - x^2}\,dx}{\frac{1}{4}\pi a^2} = \frac{4a}{3\pi}$$

Similarly, by dividing up the area into elements by lines drawn parallel to *OX*, or by symmetry, it may be seen that

$$\bar{y} = \frac{4a}{3\pi}$$

15. ***Stability*** dependent on the position of the centre of gravity.

Illustrations of stable, unstable, and neutral equilibrium.

16. When *work* is done by gravitation only, it is represented in foot-pounds or in kilogram-metres by the product of the weight of the body by the *vertical* displacement of the centre of gravity.

N.B.—Practically, it is always necessary to allow for work done by or against some kind of friction : sometimes this is so small as to be negligible, but usually it modifies considerably the theoretical relation obtained from the above statement.

17. ***Examples.***

(1) A cross is formed of six equal squares joined together: find the position of the centre of gravity.

(2) A circular lamina, 4 inches in diameter, has two holes cut out of it, one 1.5 inches in diameter and the other 1 inch in diameter, with their centres 1 inch and 1.25 inches, respectively, from the centre of the lamina, and situated on diameters mutually perpendicular. Find the centre of gravity of the portion left.

Ans : 0.202 inches from the centre.

(3) If two intersecting tangents be drawn from the extremities of a circle 4 feet in diameter, find the distance of the centre of gravity of the area inclosed between the tangents and the arc from either tangent.

Ans. 5.36 inches.

(4) A solid of uniform material consists of a cylinder 4 inches in diameter and 10 inches in length, with a hemispherical end, the circular face of which coincides with one end of the cylinder. The other end of the cylinder is pierced by a cylindrical hole, 2 inches in diameter, extending to a depth of 7 inches along the cylinder and co-axial with it. Find the centre of gravity of the solid.

Ans. 6.08 inches from the flat end.

(5) A solid consists of a hemisphere and a cylinder, each 10 inches in diameter, the centre of the base of the hemisphere being at one end of the axis of the cylinder. What is the greatest length of cylinder consistent with stability, when the solid rests with its curved end on a horizontal plane?

Ans. $\frac{5}{\sqrt{2}}$ inches.

(6) A uniform chain, 40 feet long, and weighing 10lbs. per foot, hangs vertically. How much work is necessary to wind it up.

Ans. 8000 foot pounds.

(7) A cubical block of stone, whose edge is 3 feet, rests with one face on the ground: the material weighing 150 lbs. per cubic foot. How much work is necessary to tilt the block into a position of unstable equilibrium resting on one edge?

Ans. 2514 foot pounds.

(8) If $ABCD$ be a tetrahedron, and if a plane CDE, passing through the edge CD, cuts AB in E, prove that the line joining the centres of gravity of the tetrahedra $ABCD$, $AECD$ is parallel to AB.

(9) If a right-angled triangular lamina be suspended from one of the points of trisection of the hypothenuse, shew that it will rest with one side horizontal.

18. Relation between *Work* and *Kinetic Energy*.

The total work done on a body or on a system of bodies is equal to the increase or diminution of kinetic energy: work being considered as either positive or negative.

19. Energy of a body rotating about an axis.

If a body be rotating about an axis and, at any instant, has an angular velocity ω, then its kinetic energy *at that instant* is equal to $\frac{1}{2}\Sigma(mr^2\omega^2)$.

This is evident, since each small mass m, at a distance r from the axis of rotation, has a linear velocity $r\omega$, and therefore energy represented by $\frac{1}{2} m r^2 \omega^2$.

This expression may be put in the form $\frac{1}{2} \omega^2 . \Sigma(m r^2)$.

$\Sigma(m r^2)$ is termed a *moment of inertia* and is denoted by the letter I.

Thus the energy is equal to $\frac{1}{2} I \omega^2$.

If the body be rotating uniformly then the energy remains constant. If not, then ω is continually changing; but the expression for the energy at any given instant still assumes the form given, ω being always the angular velocity at the time in question.

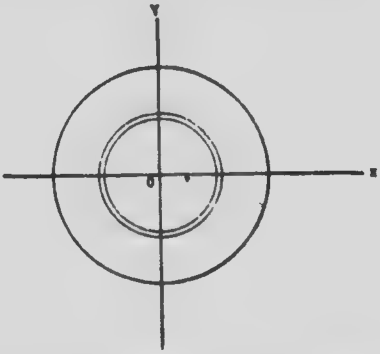
20. *Moments of Inertia.*

It will thus be seen that the energy of a rotating body may be calculated if we know its angular velocity and its moment of inertia.

The determination of the former is simple and involves only arithmetical computation; but the determination of a moment of inertia involves the summation of a series of infinitely small terms: and this involves the methods of the integral calculus.

The following illustrations will serve to shew how these moments of inertia may be most easily determined.

21. A circular plate, radius a, revolves about an axis through its centre of gravity perpendicular to the plate, with uniform angular velocity (ω): to calculate its moment of inertia and kinetic energy.



Divide the circular area up into annuli, each of infinitely small width Δr, of which one is shewn in the figure, of radius r.

Each portion of this annulus is at the same distance r from the axis of rotation, which is perpendicular to the plate through O.

The moment of inertia of this annulus is therefore equal to $m \times 2\pi r \Delta r \times r^2$, the density being taken as m. Therefore the total moment of inertia of the whole plate is

$$\Sigma(2\pi r \Delta r \times r \times m^2) = m \,.\, \Sigma(2\pi r^3 \Delta r)$$

$$= m\int_0^a 2\pi r^3 \, dr = 2\pi m \int_0^a r^3 \, dr = 2\pi m \,.\, \frac{a^4}{4}$$

$$= \pi a^2 \,.\, m \,.\, \frac{a^2}{2}$$

$$= M \,.\, \frac{a^2}{2}$$

If M be the mass of the plate.

The energy of rotation is equal to

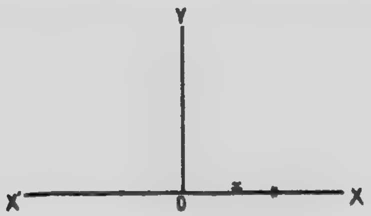
$$\tfrac{1}{2}\Sigma(m r^2 \omega^2) = \tfrac{1}{2}\omega^2 \,.\, I$$

$$\text{But } I = M \,.\, \frac{a^2}{2}$$

$$\therefore \text{ energy} = \tfrac{1}{2} \,.\, \omega^2 \,.\, M \,.\, \frac{a^2}{2}$$

$$= \omega^2 M \frac{a^2}{4} = M \,.\, \frac{a^2 \omega^2}{4}.$$

22. A straight uniform rod rotates about an axis through its centre, perpendicular to itself, with uniform angular velocity, ω: to determine the moment of inertia and the kinetic energy.



Let the rod, XX', be divided up into an infinite number of infinitely small elements, each equal to $m \Delta x$, the density being m.

Then the moment of inertia of the whole rod about OY is equal to

$$\Sigma(m\,\Delta x \times x^2)$$

This being calculated for both sides of the origin.

Hence

$$I = 2\int_0^a m\,x^2\,dx$$

If the length of the rod be $2a$.

$$\therefore I = 2m.\,\frac{a^3}{3} = M.\,\frac{a^2}{3}$$

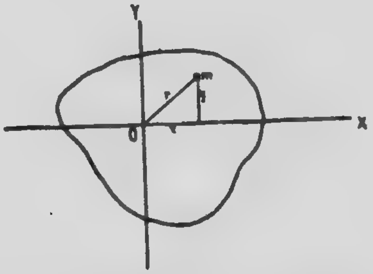
If M be the mass of the rod.

Therefore the kinetic energy is equal to

$$\tfrac{1}{2}\omega^2.\,\Sigma m r^2 = \tfrac{1}{2}\omega^2.\,M\frac{a^2}{3} = \omega^2.\,M\frac{a^2}{6}$$

23. Although moments of inertia may be always found by the process of integration, yet much labour may be saved by the following propositions.

Proposition I



Let there be any *thin plate*, of any shape, as shown in the figure, and let O be any point in its plane; OX, OY, two rectangular axes.

Then the moment of inertia of the plate about an axis through O perpendicular to the plate is equal to $\Sigma(mr^2)$, where m is any small mass of the plate situated at the point x, y.

But since $r^2 = x^2 + y^2$, we have

$$\Sigma m r^2 = \Sigma m x^2 + \Sigma m y^2.$$

That is, the moment of inertia about the line through O perpendicular to the plate is equal to the sum of the moments about the two rectangular axes. This proposition may be expressed in the form

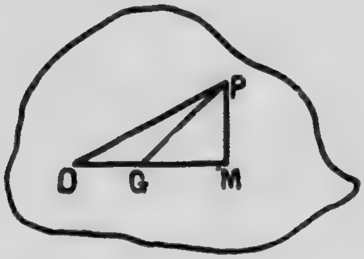
$$I = I_y + I_x.$$

Proposition II.

If, for any body, I be the moment of inertia about any axis, and I_G be the moment of inertia about a parallel axis through the centre of gravity, and p be the perpendicular distance between the two axes, then

$$I = I_G + Mp^2$$

where M is the mass of the body.



Take any section through the body perpendicular to the two axes, and let this section contain the centre of gravity of the body. Then, if P be any point in this section, join OP, GP, and complete the figure: G being the centre of gravity and $OG = p$.

Then the moment of inertia about the axis through O, perpendicular to the section is equal to $\Sigma(m \times OP^2)$.

But $OP^2 = OG^2 + GP^2 + 2OG.GM$.

$$\therefore \Sigma(mOP^2) = \Sigma(mOG^2) + \Sigma m(GP^2) + 2OG.\Sigma(m.GM)$$

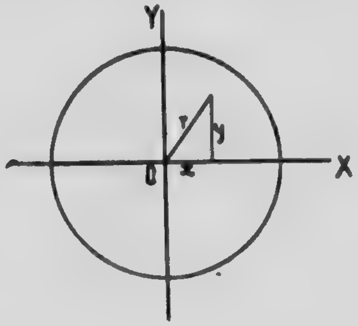
$$\therefore I = I_G + Mp^2$$

Since $\Sigma(m \times GM) = O$, by definition of the centre of gravity.

N.B.—What is true of the section through the centre of gravity is true also of all parallel sections, and therefore the above formula is true of the entire body.

24. *Illustrative Examples.*

(1) Find the moment of inertia of a circular plate about any diameter.



Since the moment of inertia of the plate about an axis through O perpendicular to the plate is $M\frac{a^2}{2}$, we have

$$M\frac{a^2}{2} = \Sigma m r^2 = \Sigma m x^2 + \Sigma m y^2$$

when r is the distance from O to any point x, y.

But $\Sigma m x^2 = \Sigma m y^2$, by symmetry;

$$\therefore \Sigma m x^2 = \Sigma m y^2 = \tfrac{1}{2}.M\frac{a^2}{2} = M\frac{a^2}{4}$$

(2) Find the moment of inertia of a circular plate about a tangent at any point on the edge.

If I be the required moment of inertia and I_G the moment of inertia about a parallel line through the centre of the plate,

$$\therefore I = I_G + M a^2$$

where a is the radius of the circle.

$$\text{And } I_G = M\frac{a^2}{4}$$

$$\therefore I = M\frac{a^2}{4} + M a^2 = M.\frac{5a^2}{4}$$

(3) The moment of inertia of a circular plate about a line through any point of the edge, perpendicular to the plate is

$$M\frac{a^2}{2} + M a^2 = M\frac{3a^2}{2}$$

(4) The moment of inertia of a rectangular plate, sides $2a$, $2b$, about a line through the centre, perpendicular to the sides ($2a$), is equal to $M\frac{b^2}{3}$; about a line through the centre, perpendicular to the sides ($2b$), is equal to $M\frac{a^2}{3}$; and about a line through the centre, perpendicular to the plate itself, is $M\frac{a^2+b^2}{3}$

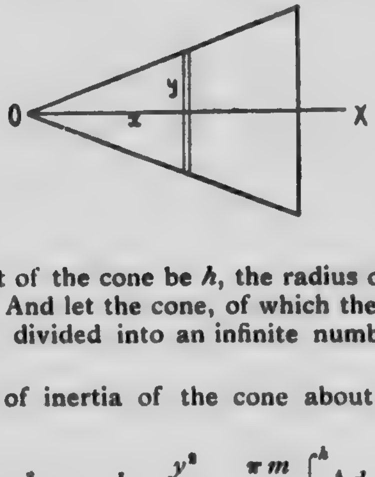
Also, the moments of inertia of the plate about its two edges are respectively

$$M\frac{4a^2}{3},\ M\frac{4b^2}{3}$$

For a square plate, $a=b$.

The above results are obtained by dividing the plate into an infinite number of infinitely small rods.

(5) To find the moment of inertia of a solid right circular cone about its axis.



Let the height of the cone be h, the radius of the base b, and the density m. And let the cone, of which the section is shewn in the figure, be divided into an infinite number of slices each of width Δx.

The moment of inertia of the cone about its axis, OX, is equal to

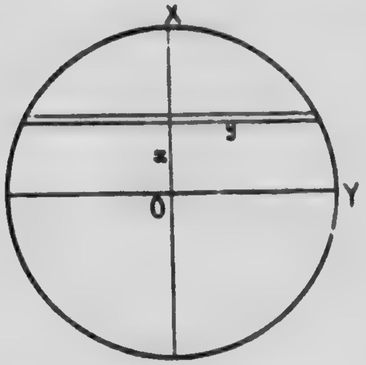
$$\int_0^h \pi y^2 . m . dx . \frac{y^2}{2} = \frac{\pi m}{2}\int_0^h y^4\, dx$$

$$\text{But } \frac{y}{x} = \frac{b}{h} \text{ and } \therefore y = \frac{b}{h}x$$

$$\therefore I = \frac{\pi m}{2}\cdot\frac{b^4}{h^4}\int_0^h x^4\, dx$$

$$= M \cdot \tfrac{3}{10} b^2.$$

6. The moment of inertia of a sphere, of radius a, about any diameter.



Let the sphere be divided into an infinite number of slices, perpendicular to the diameter OX, each of thickness Δx.

Then

$$I\,(\text{about } OX) = 2\int_0^a \pi y^2 m\,dx\,.\,\frac{y^2}{2}$$

m being the density.

$$\therefore I = \pi m \int_0^a y^4\,dx = \pi m \int_0^a (a^2 - x^2)^2\,dx$$

Since $y^2 + x^2 = a^2$

$$\therefore I = M\,.\,\tfrac{2}{5}\,a^2.$$

7. The moment of inertia of an elliptic plate, whose axes are $2\,a$, $2\,b$, about the major axis is $M\dfrac{b^2}{4}$; about the minor axis $M\dfrac{a^2}{4}$; and, about an axis through the centre, perpendicular to the plate, $M\dfrac{a^2 + b^2}{4}$.

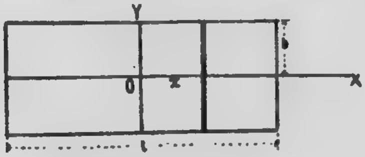
This is deduced from the case of a circular plate.

8. The moment of inertia of a hemisphere about any diameter of its base is evidently half what it would be for the complete sphere and is therefore $M\,.\,\tfrac{2}{5}\,a^2$ where M is the mass of the hemisphere.

9. A spherical shell of radii R, r about any diameter.

$$I = M \cdot \frac{2}{5} \frac{R^5 - r^5}{R^3 - r^3}.$$

10. The moment of inertia of a circular cylinder, of length l and radius b, about its axis is $M \frac{b^2}{2}$; and about a line through its centre of gravity perpendicular to the axis is $\frac{M}{4}\left(\frac{l^2}{3} + b^2\right)$.



The first result follows from the moment of inertia of a circular plate about an axis through its centre of gravity perpendicular to the plate.

The second result is obtained as follows:

Let the cylinder be divided into slices, parallel to the given line, and perpendicular to the axis of the cylinder.

Then I (about OY)

$$= 2\int_0^{\frac{l}{2}} \pi b^2 m \, dx \left(\frac{b^2}{4} + x^2\right)$$

$$= 2\pi b^2 m \int_0^{\frac{l}{2}} \left(\frac{b^2}{4} + x^2\right) dx$$

$$= M\left(\frac{b^2}{4} + \frac{l^2}{12}\right)$$

m being the density of the cylinder.

11. A plane figure consists of a rectangle 8 inches by 4 inches, with a rectangular hole 6 inches by 2 inches, cut so that the diagonals of the two rectangles are in the same straight lines.

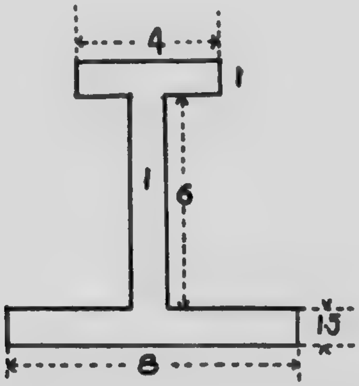
Find the moment of inertia of the figure and the radius of gyration about one of the short outer sides.

N.B.—If $\Sigma m r^2 = M k^2$, then k is termed a *radius of gyration.*

In this particular case $k = 4.93$ inches.

12. Find the moment of inertia of the area inclosed between two concentric circles of 10 inches and 8 inches diameter respectively, about a diameter of the circles.

13.



Shew that the radius of gyration of this section about the lowest horizontal edge is 4.29 inches.

14. The moment of inertia of a grindstone 3 feet in diameter and 8 inches thick about its axis of rotation is 709.3 (lb – ft^2) units, the density of the material being 2·14.

15. The kinetic energy of the stone in the preceding example, rotating with an angular velocity of 5 revolutions in 6 seconds, is 303.7 foot pounds.

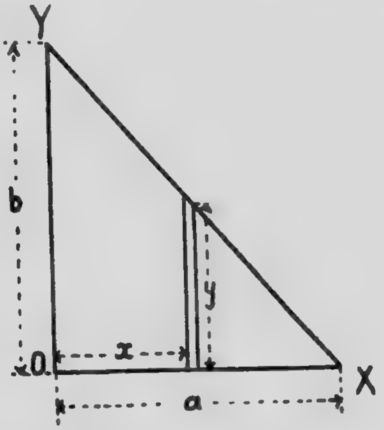
•To solve this we have :

$$
\begin{aligned}
E &= \tfrac{1}{2} I \omega^2 \\
&= \tfrac{1}{2} \times 709.3 \times \left(\frac{5 \times 2\pi}{6}\right)^2 \div 32.
\end{aligned}
$$

16. A door 7.5 feet high and 3 feet wide, weighing 80 lbs, swings on its hinges so that the outward edge moves at the rate of 8 feet per second. How much work must be expended in stopping it ?

$$E = \tfrac{1}{2} I \omega^2 = \tfrac{1}{2} \times \frac{80}{32} \times \frac{2.25 \times 4}{3} \times \left(\frac{8}{3}\right)^2 = 26.7 \text{ ft. pounds.}$$

17. To find the moment of inertia of a right angled triangular area about either side.



Divide the area into strips of width Δx. Then, if the sides be a, b, and the density be m,

$$I \text{ about } OY = \int_0^a m\, y\, dx \,.\, x^2.$$

But, from the figure, $y = \dfrac{b}{a}(a - x)$

$$\therefore I = \frac{mb}{a}\int_0^a (a\,x^2 - x^3)\, dx = M \,.\, \frac{a^2}{6}$$

if M be the mass of the triangular area.

Similarly it may be shewn that the moment of inertia about OX is $M. \dfrac{b^2}{6}$.

Also, the moment of inertia about an axis through O, perpendicular to the plate, is $M\left(\dfrac{a^2 + b^2}{6}\right)$.

(18) The moment of inertia of a square plate about a diagonal is $M\dfrac{a^2}{3}$, if the side of the plate be $2a$.

25. *Principal Axes:* at any point of a body, and particularly at the centre of gravity.

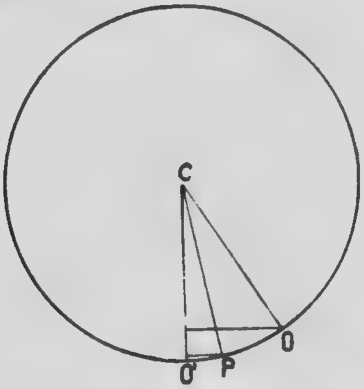
Illustrations of the meaning of this term.

26. *Harmonic motion.*

General explanation. Illustrations of this kind of motion: tuning-fork, pendulum, vibrating spring, oscillating spring, etc.

27. *The Simple Pendulum.*

This consists of a fine string, whose mass is supposed to be negligible. One end is fixed, and to the other end is attached a small heavy mass which is allowed to move to and fro under the action of gravity.



Let CO be a simple pendulum, of length l, and let it start from θ, where the angle $OCO' = \alpha$.

When it reaches the point P, under the action of gravity, where $PCO' = \theta$, we have, by the equivalence of work and energy, the following relation:

$$\tfrac{1}{2} m v^2 = m g \{(l - l \cos \alpha) - (l - l \cos \theta)\}$$

$$\therefore v^2 = 2 g l \{\overline{1 - \cos \alpha} - \overline{1 - \cos \theta}\} = 2 g l (\cos \theta - \cos \alpha)$$

$$\therefore v = \sqrt{2 g l} \,.\, \sqrt{\cos \theta - \cos \alpha}$$

which determines the linear velocity at any point, when the corresponding angles θ, α, are given.

The velocity at the lowest point O' is given by

$$v = \sqrt{2 g l}\ \sqrt{1 - \cos \alpha} = 2 \sin \frac{\alpha}{2} \sqrt{g l}\,.$$

If α, θ both be very small, then the relation becomes

$$v^2 = 2 g l \left(2 \sin^2 \frac{\alpha}{2} - 2 \sin^2 \frac{\theta}{2}\right)$$

$$\therefore v^2 = g l (\alpha^2 - \theta^2)$$

$$v = - \sqrt{g l}\ \sqrt{\alpha^2 - \theta^2}$$

the negative sign being taken to indicate that the motion is towards O'.

This gives the actual velocity at the point P, and hence, if Δt be a small increment of time after the pendulum passes the point P, and $l\Delta\theta$ be the corresponding small distance traversed, we must have

$$l\,\Delta\,\theta = -\sqrt{g\,l}\,\sqrt{\alpha^2-\theta^2}\,.\,\Delta\,t$$

and therefore

$$\Delta\,t = -\frac{l\,\Delta\,\theta}{\sqrt{g\,l}\,\sqrt{\alpha^2-\theta^2}}$$

$$= -\sqrt{\frac{l}{g}}\cdot\frac{\Delta\,\theta}{\sqrt{\alpha^2-\theta^2}}$$

Consequently the total time from O to O', when the angles α, θ are small, is

$$-\int_{\alpha}^{0}\sqrt{\frac{l}{g}}\,\frac{d\theta}{\sqrt{\alpha^2-\theta^2}} = \sqrt{\frac{l}{g}}\int_{0}^{\alpha}\frac{d\theta}{\sqrt{\alpha^2-\theta^2}}$$

$$= \frac{\pi}{2}\sqrt{\frac{l}{g}}\,.$$

And, since the pendulum after reaching O' evidently has sufficient energy to carry it to O'', such that $O'O'' = OO'$, and will therefore *oscillate* between O and O'', we must have the total time of a complete oscillation given by

$$T = 4\times\frac{\pi}{2}\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{l}{g}}\,.$$

28. Experimental proof of the truth of this formula. Practical applications. Determination of g. Variation of value of g at different points of earth's surface.

29. *Illustrative Examples.*

(1) Find the length of a seconds pendulum at Toronto, where $g = 32.08$.

(2) A seconds pendulum is found to lose 48.6 seconds in a day at the summit of a mountain; find the height of the mountain.

Ans. 2.25 miles.

(3) Find the length of a pendulum which beats half-seconds at Greenwich ($g = 32.19$).

Ans. 9.78 inches.

(4) The length of the second pendulum at the equator is 39.014 ins.; find the local acceleration of gravity.

Ans. $g = 32.075$

30. *The Compound Pendulum.*

Any body capable of oscilliating about a fixed horizontal axis, under the action of gravity, may be termed a compound pendulum.

Referring to the figures of Article 27, it may be seen that if we represent the body by G, its centre of gravity, at a distance h from the fixed axis, the equation of energy becomes

$$\tfrac{1}{2}\Sigma mv^2 = Mgh\left\{\cos\theta - \cos a\right\}$$

M being the total mass of the body, and h replacing l in the circle shewn.

$$\text{But } v = r\omega$$

$$\therefore\ \tfrac{1}{2}\Sigma mr^2\omega^2 = Mgh\left\{\cos\theta - \cos a\right\}$$

When θ, a, are both small, this becomes

$$\omega^2 = \frac{gh}{h^2+k^2}(a^2-\theta^2)$$

which gives the angular velocity at any time. Following a similar line of reasoning to that of the former article it may be shewn that the time of a complete small oscillation is

$$2\pi\sqrt{\frac{h^2+k^2}{gh}}$$

31. The formula for the time of a small oscillation of a compound pendulum may be written $2\pi\sqrt{\dfrac{l}{g}}$, where the value of l is $\dfrac{h^2+k^2}{h} = h + \dfrac{k^2}{h}$: this is called the length of the *equivalent simple pendulum.*

32. Experimental verification of the preceding formula :

(1) Rod moving in a vertical plane, about one end fixed, under gravity

(2) C [illegible] plate, oscillating about an axis through the rim perpenc [illegible] r to the plane of the plate.

In these two cases, from the equation of energy, we may find the angular velocity in any position ; and, also, determine the time of a small oscillation.

$$\text{For the rod, } T = 2\pi\sqrt{\frac{4a}{3g}}$$

$$\text{For the plate, } T = 2\pi\sqrt{\frac{3a}{2g}}$$

33. Since $\omega^2 = \frac{2gh}{h^2+k^2}(\cos\theta - \cos a)$

we get, by differentiation, with respect to t the time,

$$2\omega\frac{d\omega}{dt} = -\frac{2gh}{h^2+k^2}\sin\theta\frac{d\theta}{dt}$$

$$\text{But } \omega = -\frac{d\theta}{dt}$$

since the angular velocity is continually changing at every instant, and is negative since θ diminishes as t increases.

$$\therefore \frac{d\omega}{dt} = \frac{gh}{h^2+k^2}\sin\theta$$

and $\frac{d\omega}{dt}$ is the angular acceleration (f)

$$\therefore M(h^2+k^2)f = Mgh\sin\theta$$

$$\text{or } If = \text{Torque}$$

Since $I = M(h^2+k^2)$, and $Mgh \sin\theta$ is the *torque* or turning moment of the force of gravity about the axis of rotation.

34. *Centre of suspension* and *centre of oscillation.*

For every body capable of rotation about a horizontal axis, under gravity, there exists two points which are reversible, when we consider the time of a small oscillation.

Illustrate by reference to a straight uniform rod, furnished with two knife edges, one at the end and the other at the centre of oscillation $\left(\frac{4a}{3}\right)$

35. *The Torsion Pendulum.*

Illustrate experimentally the fact that the time of oscillation varies as the square root of the moment of inertia: circular plate, rectangular plate, cylinder.

36. *Balance Wheel* of watch or clock.

When the torsion wire is replaced by a steel spring, the formula $T \propto \sqrt{I}$ still holds.

Illustrate by means of model with different rings and masses which alter the moment of inertia about the axis of rotation.

37. In general, when any body rotates about an axis under the action of external force, so long as this axis does not pass

through the centre of gravity, the time of rotation from rest will be determined from the equation of energy

$$\tfrac{1}{2} I \omega^2 = W$$

W being the work done

$$\therefore \omega = \sqrt{\frac{2W}{I}}$$

$$\therefore \Delta\theta = \sqrt{\frac{2W}{I}}\,\Delta t$$

$$\therefore \Delta t = \sqrt{I}\cdot\frac{\Delta\theta}{\sqrt{2W}}$$

$$\therefore t = \sqrt{I}\int\frac{d\theta}{\sqrt{2W}}$$

that is, the time to any particular position varies as the square root of the moment of inertia, whatever be the relation (supposed finite and real) between W and θ.

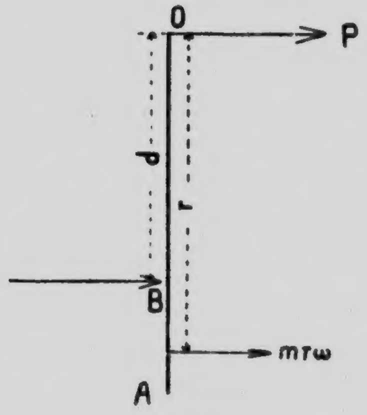
38. *Impulsive Forces.*

An impulsive force is a very great force acting for a very short time, and may be measured by a change of momentum.

If, for example, a body be moving in a straight line with a velocity v, and this be changed suddenly to v', *in the same straight line*, then the value of the impulsive force which produced the change of velocity is $m(v' - v)$ where m is the mass of the body and v' is supposed to be greater than v.

Shew experimentally the difference between a finite and impulsive force.

39. *Action of an impulsive force* on a rod hanging freely from one end.



Let the rod OA be struck a blow, B, at a point distant d from the axis of rotation, and in consequence of the action of this blow, let each elemental mass of the rod have a momentum $mr\omega$, where ω is the angular velocity produced suddenly by the blow.

Then evidently there will be some kind of impulsive pressure on O, which we may denote by P.

Then we must have

$$B = P + \Sigma(mr\omega) \qquad (1)$$

and if, in addition, we assume that the principle of moments applies to impulsive forces, we get

$$B \times d = \Sigma(mr^2\omega) = \omega \,.\, \Sigma(mr^2) \qquad (2)$$

From these two equations it will be seen that

$$\omega = \frac{3Bd}{4Ma^2}$$

and $B = P + Ma\omega$

$$\therefore P = B - Ma\omega$$

$$= B - \frac{3Bd}{4a}$$

$$\therefore P = B\left\{1 - \frac{3d}{4a}\right\}$$

and therefore P will be positive, negative, or zero, according as

$$d < > = \frac{4a}{3}\,.$$

40. Illustrate the preceding relation by means of experiments.

41. When $d = \frac{4a}{3}$, there is no impulsive pressure on the axis through O, and the point at which the blow is struck is called the *centre of percussion.*

42. Bodies, capable of motion about an axis, and symmetrical with respect to a line, have a centre of percussion on this line distant $\frac{k^2 + h^2}{h}$ from the axis: where h is the distance from the centre of gravity to the axis and k is the radius of gyration.

43. Composition of angular velocities.

The top and gyroscope.